

Theta functions on Kodaira–Thurston manifold ^{*}

D.V. Egorov

Abstract

We define analogue of theta functions on the Kodaira–Thurston manifold which is a compact 4-dimensional symplectic manifold and use them to construct canonical symplectic embedding of the Kodaira–Thurston manifold into the complex projective space (analogue of the Lefschetz theorem).

Keywords: theta functions, Kodaira–Thurston manifold, symplectic embedding

1 Introduction

In this work we construct analogue of classical theta functions on the Abelian variety for the Kodaira–Thurston nilmanifold M_{KT} . The classical theta function from the geometrical point of view is a section of a holomorphic line bundle over complex torus. The theta function on the Kodaira–Thurston manifold is defined as section (not holomorphic) of a special line bundle L over M_{KT} .

Analogues of theta functions for nilmanifolds were defined earlier [1, 2], but all these generalizations are based on representation theory. We construct such *analogue of theta functions with characteristics that they set canonical symplectic embedding of M_{KT} into a complex projective space (analogue of Lefschetz theorem)*.

The Kodaira–Thurston manifold M_{KT} is the quotient of \mathbb{R}^4 by the free action of discrete group Γ , which generators are

$$\begin{aligned} a : (x, y, z, t) &\rightarrow (x + 1, y, z + y, t) \\ b : (x, y, z, t) &\rightarrow (x, y + 1, z, t) \\ c : (x, y, z, t) &\rightarrow (x, y, z + 1, t) \\ d : (x, y, z, t) &\rightarrow (x, y, z, t + 1) \end{aligned} \tag{1}$$

^{*}This work is supported by RFFI, grant 06-01-00094-a.

The Kodaira–Thurston manifold is notable as the first known example of a symplectic but not Kähler manifold [3].

Note that the embedding of M_{KT} into \mathbb{CP}^n can't be holomorphic, since M_{KT} is not Kähler. However we prove that this map is symplectic. In other words the Fubini–Study form on \mathbb{CP}^n induces symplectic structure on M_{KT} .

In §2 we recall necessary facts from classical theta function theory, in §3 we give the definition of theta function on M_{KT} and study some of its properties, in §4 we construct embedding into complex projective space (Theorem 1) and in §5 prove that this embedding is symplectic (Theorem 2).

Author is grateful to I.A. Taimanov for posing the problem and to A.E. Mironov for helpful discussions.

2 Classical theta function

Let's recall some known facts about the classical theta function of Jacobi on one-dimensional complex torus. We will need them in the sequel.

Consider formal series

$$\theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{2\pi i k z + \pi i k(k-1)\tau}.$$

If $\text{Im } \tau > 0$ then this series converges uniformly on every compact subset of \mathbb{C} and thus define entire function. In notations of [4] this theta function is written as

$$\exp(-\pi i \tau / 4 + \pi i z) \cdot \theta_{-1/2, 0}(z, \tau).$$

The invariance under map $\tau \rightarrow \tau + 1$ and certain periodicity conditions explain our particular choice

$$\theta(z + 1, \tau) = \theta(z, \tau), \tag{2}$$

$$\theta(z + \tau, \tau) = \exp(-2\pi i z) \cdot \theta(z, \tau). \tag{3}$$

The generalization of theta function is the theta function of degree $k \in \mathbb{N}$. It is an entire function $\theta_k(z, \tau)$ which periodicity conditions are as follows

$$\theta_k(z + 1, \tau) = \theta_k(z, \tau),$$

$$\theta_k(z + \tau, \tau) = \exp(-2\pi i k z) \theta_k(z, \tau).$$

It is not hard to prove that theta functions of degree k form linear space of dimension k . Let's denote this space by \mathcal{L}_k .

The product of theta functions can be the theta function of higher degree. Let $\{\alpha_i\}_{i=1}^k$ be the set of constants such that the sum of them is equal to zero. Then the following product is theta function of degree k :

$$\prod_{i=1}^k \theta(z + \alpha_i, \tau) \in \mathcal{L}_k.$$

Theta function is equal to zero at the point $z = 1/2$. There is a single up to multiplicity zero in the fundamental region of lattice $\mathbb{Z} + \tau\mathbb{Z}$.

Theta function $\theta(z, \tau)$ satisfies following PDE:

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{4\pi i} \cdot \frac{\partial^2 \theta}{\partial z^2} - \frac{\theta}{2} \quad (4)$$

Let $\{\theta_k^p(z, \tau)\}_{p=1}^k$ be the basis in the space of theta functions of degree k . Then map written in homogenous coordinates

$$\varphi_k(z) = [\theta_k^1(z, \tau) : \dots : \theta_k^k(z, \tau)]$$

is well-defined map of complex torus into \mathbb{CP}^{k-1} .

The Lefschetz theorem holds: *if $k \geq 3$ then map φ_k is embedding*. Note that this theorem is true for Abelian tori of arbitrary dimensions.

3 Definition of theta function on the Kodaira–Thurston manifold

Projection map $(x, y, z, t) \rightarrow (y, t)$ gives M_{KT} the structure of T^2 -bundle over T^2 . Left-invariant symplectic form $\omega_{KT} = (dz - xdy) \wedge dx + dy \wedge dt$ tames bundle structure. This means that restriction of ω_{KT} on any fibre and base is not degenerated.

Hence let the space of theta functions of degree k on M_{KT} be the span of products of classical theta functions on fibre and base:

$$\theta_k^p(z + ix, y + i) \cdot \theta_k^q(y + it, i); \quad p, q = 1, \dots, k$$

Let's denote this space by \mathcal{L}_k . Clearly dimension of \mathcal{L}_k is equal to k^2 .

Theta function of degree one we denote as

$$\theta_{KT}(x, y, z, t) = \theta(z + ix, y + i) \cdot \theta(y + it, i).$$

3.1 Theta function — section of complex line bundle

The periodicity conditions of theta function on M_{KT} are as follows:

$$\begin{aligned}\theta_{KT}(x+1, y, z+y, t) &= \exp(-2\pi i(z+ix)) \cdot \theta_{KT}(x, y, z, t) \\ \theta_{KT}(x, y+1, z, t) &= \theta_{KT}(x, y, z, t) \\ \theta_{KT}(x, y, z+1, t) &= \theta_{KT}(x, y, z, t) \\ \theta_{KT}(x, y, z, t+1) &= \exp(-2\pi i(y+it)) \cdot \theta_{KT}(x, y, z, t)\end{aligned}\tag{5}$$

These formulae imply that θ_{KT} is a section of line bundle over M_{KT} . This bundle is obtained by factorization $\mathbb{R}^4 \times \mathbb{C}$ under the action of group Γ

$$(u, w) \sim (\lambda \cdot u, e_\lambda(u)w), \quad u \in \mathbb{R}^4, w \in \mathbb{C}, \lambda \in \Gamma$$

where $e_\lambda(u)$ are multipliers i.e. nonzero functions

$$e_\lambda : \mathbb{R}^4 \rightarrow \mathbb{C}^*$$

which satisfy following identities

$$\begin{aligned}e_\lambda(\mu \cdot u)e_\mu(u) &= e_{\lambda\mu}(u), \quad \lambda, \mu \in \Gamma \\ e_0(u) &= 1.\end{aligned}$$

Sections of bundle which is set by multipliers are in one-to-one correspondence with functions f on \mathbb{R}^4 such that

$$f(\lambda \cdot u) = e_\lambda(u)f(u), \quad \lambda \in \Gamma, u \in \mathbb{R}^4.$$

Let's note that the relations in group Γ hold for multipliers. It is obvious since multipliers are set by behavior of the same function.

3.2 Multiplicative property of θ_{KT}

Let's define the action of $\zeta = (\zeta^1, \zeta^2) \in \mathbb{C}^2$ on θ_{KT} :

$$(\zeta \cdot \theta_{KT})(x, y, z, t) = \theta(z + ix + \zeta^1, y + i)\theta(y + it + \zeta^2, i).\tag{6}$$

If

$$\sum_{i=1}^k \alpha_i = 0$$

then the following product is theta function of degree k :

$$\prod_{i=1}^k (\zeta_i \cdot \theta_{KT})(x, y, z, t) \in \mathcal{L}_k.\tag{7}$$

The proof follows from the analogous property of classical theta function.

4 Embedding of M_{KT} into complex projective space

Let's enumerate the basis of \mathcal{L}_k : $\{s_i\}_{i=1}^{k^2}$. Map

$$\varphi_k = (s_1, s_2, \dots, s_{k^2})$$

is well defined map of M_{KT} to \mathbb{CP}^{k^2-1} .

Theorem 1 *If $k \geq 3$ then map φ_k is embedding.*

PROOF. For brevity we will prove the theorem in case of $k = 3$. The proof for $k > 3$ is analogous.

Firstly we will prove the injectivity of φ_k . We will follow the proof of the classical Kodaira embedding theorem stated in [5, ch. 1, §4]

Note that theta functions of degree k are the global sections of k -th tensor power of bundle set by multipliers (5).

If for any two points $u \neq v \in M_{KT}$ there exists section $s \in \mathcal{L}_k$ such that $s(u) = 0$ and $s(v) \neq 0$ then map φ_k is injective. Indeed, assume that map "glues" points u and v . It means that for all sections $s \in \mathcal{L}_k$ it is true that $s(v) = \zeta \cdot s(u)$, where ζ is nonzero constant. If s satisfies previously mentioned condition then ζ must be zero and we have contradiction.

Note also that given condition implies that for any point $u \in M_{KT}$ not all sections vanish at point u .

We construct needed theta function of degree $k = 3$ as a product of two functions $s = fg$:

$$f(x, y, z, \alpha, \beta) = \theta(z + ix + \alpha, y + i)\theta(z + ix + \beta, y + i) \times \\ \times \theta(z + ix - \alpha - \beta, y + i), \quad (8)$$

$$g(y, t, \gamma, \delta) = \theta(y + it + \gamma, i)\theta(y + it + \delta, i)\theta(y + it - \gamma - \delta, i). \quad (9)$$

As follows from (7) product fg is a valid theta function of degree $k = 3$ on M_{KT} .

We denote coordinates of u, v as (x, y, z, t) and (x', y', z', t') respectively. Select γ such that $\theta(y + it + \gamma, i) = 0$. Now select δ such that all other factors in definition of function g don't vanish at the point v :

$$\theta(y' + it' + \delta, i)\theta(y' + it' - \gamma - \delta, i) \neq 0.$$

It is possible because zeros of theta function are isolated. Also by selecting α, β we can assure that function f doesn't vanish at point v .

Unless $\theta(y' + it' + \gamma, i) = 0$ the constructed section solves the problem. Assume contrary. Since classical theta function has single (up to multiplicity) zero in the fundamental region of lattice, it follows that $y = y'$, $t = t'$ modulo Γ .

Select α such that $\theta(z + ix + \alpha, y + i) = 0$. Note that $\theta(z' + ix' + \alpha, y' + i) \neq 0$, because otherwise $u = v$. Select β such that $f(v) \neq 0$ and γ, δ such that $g(v) \neq 0$.

Thus we constructed necessary section and proved the injectivity of φ_k .

Let's prove that rank of φ_k is maximal. Here we will follow the proof of Lefschetz theorem stated in [6]. Firstly we show that the rank of φ_k is maximal if and only if the rank of matrix J is maximal

$$J = \begin{pmatrix} s_1 & \dots & s_{k^2} \\ \partial_x s_1 & \dots & \partial_x s_{k^2} \\ \partial_y s_1 & \dots & \partial_y s_{k^2} \\ \partial_z s_1 & \dots & \partial_z s_{k^2} \\ \partial_t s_1 & \dots & \partial_t s_{k^2} \end{pmatrix}.$$

The map φ_k written in homogenous coordinates is a composition of the map $\tilde{\varphi}_k$ to \mathbb{C}^{k^2} and subsequent projection $\pi : \mathbb{C}^{k^2} \setminus \{0\} \rightarrow \mathbb{CP}^{k^2-1}$. Obviously if we cross out first row of J we get the differential of $\tilde{\varphi}_k$.

Now assume that at the point $u \in M_{KT}$ first row of J is a linear combination of other rows. It means that radius-vector of $\tilde{\varphi}_k(u)$ is collinear to the image of a certain tangent vector (to M_{KT}) at the point u . Since π projects along complex lines passing through the origin, the kernel of differential of π consists exactly of such vectors. Therefore rank of J is maximal if and only if rank of φ_k is maximal.

Let's transform matrix J to more suitable for us form. The rank of the following matrix coincides with the rank of J

$$\tilde{J} = \begin{pmatrix} s_1 & \dots & s_{k^2} \\ (\partial_y - i\partial_t)s_1 & \dots & (\partial_y - i\partial_t)s_{k^2} \\ (\partial_z - i\partial_x)s_1 & \dots & (\partial_z - i\partial_x)s_{k^2} \\ (\partial_y + i\partial_t)s_1 & \dots & (\partial_y + i\partial_t)s_{k^2} \\ (\partial_z + i\partial_x)s_1 & \dots & (\partial_z + i\partial_x)s_{k^2} \end{pmatrix}.$$

Last two rows of \tilde{J} are the Cauchy-Riemann conditions. Since sections s_j are holomorphic functions of $z + ix$, the last row of \tilde{J} vanishes.

Assume that rank of \tilde{J} (over \mathbb{C}) is less than 4 at the certain fixed point $u^* = (x^*, y^*, z^*, t^*) \in M_{KT}$. It means that there exist non-trivial constants

a, b, c, d such that

$$as_j(u^*) + \frac{b}{2}(\partial_y - i\partial_t)s_j(u^*) + \frac{c}{2}(\partial_z - i\partial_x)s_j(u^*) + \frac{d}{2}(\partial_y + i\partial_t)s_j(u^*) = 0, \\ j = 1, \dots, k^2.$$

By (7), the function

$$s(u, \mu, \nu) = (\mu \cdot \theta_{KT})(u)(\nu \cdot \theta_{KT})(u)((-\mu - \nu) \cdot \theta_{KT})(u).$$

lies in $\mathcal{L}_k(k=3)$ for any μ, ν . Hence this function is a linear combination of s_j and the identity

$$as(u^*, \mu, \nu) + \frac{b}{2}(\partial_y - i\partial_t)s(u^*, \mu, \nu) + \frac{c}{2}(\partial_z - i\partial_x)s(u^*, \mu, \nu) + \\ + \frac{d}{2}(\partial_y + i\partial_t)s(u^*, \mu, \nu) = 0. \quad (10)$$

holds. We define the linear differential operator $L = \frac{b}{2}(\partial_y - i\partial_t) + \frac{c}{2}(\partial_z - i\partial_x) + \frac{d}{2}(\partial_y + i\partial_t)$ and rewrite the last identity

$$L \log(\mu \cdot \theta_{KT})(u^*) = -a - L \log(\nu \cdot \theta_{KT})(u^*) - L \log((-\mu - \nu) \cdot \theta_{KT})(u^*). \quad (11)$$

For any u, μ there exists ν such that

$$(\nu \cdot \theta_{KT})(u)((-\mu - \nu) \cdot \theta_{KT})(u) \neq 0. \quad (12)$$

By (11)-(12), the function

$$\xi(\mu) = L \log(\mu \cdot \theta_{KT})(u^*) \quad (13)$$

is an entire function of $\mu = (\mu^1, \mu^2)$. It follows from (5) that function $\xi(\mu)$ satisfies the following periodicity conditions

$$\xi(\mu^1 + 1, \mu^2) = \xi(\mu^1, \mu^2), \quad (14)$$

$$\xi(\mu^1 + y^* + i, \mu^2) = \xi(\mu^1, \mu^2) - 2\pi ic, \quad (15)$$

$$\xi(\mu^1, \mu^2 + 1) = \xi(\mu^1, \mu^2), \quad (16)$$

$$\xi(\mu^1, \mu^2 + i) = \xi(\mu^1, \mu^2) - 2\pi ib. \quad (17)$$

Therefore derivatives $\partial_{\mu^j} \xi$ are the periodic and entire functions. This means that they are constants and $\xi = \alpha \mu^1 + \beta \mu^2 + \gamma$. By (14), (16), $\alpha = \beta = 0$ and

function ξ is constant. By (15),(17), $b = c = 0$. Then the following identity holds

$$\xi(\mu) = \frac{d}{2} \left[\frac{\partial_y \theta(z + ix + \mu^1, y + i)}{\theta(z + ix + \mu^1, y + i)} \right]_{u=u^*} = \gamma. \quad (18)$$

Here we used the Cauchy-Riemann equation implicitly

$$(\partial_y + i\partial_t)\theta(y + it, i) = 0.$$

Let's denote by D the $z + ix$ -differentiation

$$D = \frac{1}{2}(\partial_z - i\partial_x).$$

It follows from (4) that

$$\partial_y \theta(z + ix, y + i) \equiv \frac{1}{4\pi i} (D^2 \theta)(z + ix, y + i) - \frac{1}{2} (D\theta)(z + ix, y + i). \quad (19)$$

By substituting (19) in (18) and considering that

$$(D\theta)(z + ix + \mu^1, y + i) = \partial_{\mu^1} \theta(z + ix + \mu^1, y + i),$$

we have that function $\theta(z^* + ix^* + \mu^1, y^* + i)$ as a function of μ^1 satisfies the linear ODE with constant coefficients

$$\frac{d}{4\pi i} \theta'' - \frac{d}{2} \theta' - 2\gamma \theta = 0.$$

It is easy to write down general solution of this equation and check that it contradicts to the periodicity conditions of theta function (5). Thus $d = \gamma = 0$. By (10), $a = 0$.

As a result all constants a, b, c, d vanish and matrix \tilde{J} has the maximal rank. Since point u^* is arbitrary, the rank is maximal everywhere. Theorem proved.

REMARK. It would be interesting to investigate the connection of this theta function to certain nonlinear equations by the way of obtaining soliton equations from secant identities for Jacobians [6].

5 Embedding is symplectic

Manifold M_{KT} is a symplectic manifold, where symplectic form would be for instance the following left-invariant form $\omega_{KT} = (dz - xdy) \wedge dx + dy \wedge dt$. In this section we will prove the following

Theorem 2 1. If $k \geq 3$ then map φ_k induces a symplectic form on M_{KT} .

2. Induced symplectic form is cohomologous to $k \cdot \omega_{KT}$.

PROOF. Let's choose the following basis of \mathcal{L}_k in the definition of φ_k

$$\theta_k^p(z + ix, y + i) \cdot \theta_k^q(y + it, i); \quad p, q = 1, \dots, k.$$

Notice that φ_k is a composition of two maps. The first one is $\psi_k : M_{KT} \rightarrow \mathbb{CP}^{k-1} \times \mathbb{CP}^{k-1}$, $\psi_k = (\psi'_k, \psi''_k)$. Here

$$\psi'_k(x, y, z) = [\theta_k^1(z + ix, y + i) : \dots : \theta_k^k(z + ix, y + i)],$$

$$\psi''_k(y, t) = [\theta_k^1(y + it, i) : \dots : \theta_k^k(y + it, i)].$$

The second is the Segre map $\sigma_k : \mathbb{CP}^{k-1} \times \mathbb{CP}^{k-1} \rightarrow \mathbb{CP}^{k^2-1}$, which is defined in homogenous coordinates by the formula

$$\sigma_k \left([z^1 : \dots : z^k], [w^1 : \dots : w^k] \right) = [z^1 w^1 : z^1 w^2 : \dots : z^k w^{k-1} : z^k w^k].$$

Thus $\varphi_k = \sigma_k \circ \psi_k$. Let's denote by Ω' symplectic form (associated with the Fubini–Study metric) on the the first factor of $\mathbb{CP}^k \times \mathbb{CP}^k$, by Ω'' on the second. Then $\Omega' + \Omega''$ is a symplectic form on the product. Since the Segre map is a holomorphic embedding, it is sufficient to show that induced form $\psi_k^*(\Omega' + \Omega'')$ is symplectic.

Note that algebra of left-invariant forms on M_{KT} is generated by $dx, dy, dz - xdy, dt$.

Map ψ''_k is holomorphic embedding of complex torus into \mathbb{CP}^k described by the classical Lefschetz theorem. Therefore the Fubini–Study form induces symplectic form on torus

$$(\psi''_k)^*(y, t)(\Omega'') = \alpha \cdot dy \wedge dt,$$

where α doesn't vanish anywhere on M_{KT} .

Let

$$(\psi'_k)^*(x, y, z)(\Omega') = f \cdot (dz - xdy) \wedge dx + g \cdot (dz - xdy) \wedge dy + h \cdot dx \wedge dy.$$

Here f, g, h are certain functions on M_{KT} . This is the general view of 2-form on M_{KT} generated by map depending on x, y, z .

Note that for any fixed y map ψ'_k also becomes holomorphic embedding described by the Lefschetz theorem and therefore

$$(\psi'_k)^*(\Omega') = \beta \cdot dz \wedge dx,$$

where β doesn't vanish anywhere on M_{KT} . It follows that $f \equiv \beta$. Gathering altogether

$$\begin{aligned} (\psi_k^*(\Omega' + \Omega''))^2 &= ((\psi'_k)^*(\Omega') + (\psi''_k)^*(\Omega''))^2 = \\ &= (\beta \cdot (dz - xdy) \wedge dx + g \cdot (dz - xdy) \wedge dy + h \cdot dx \wedge dy + \alpha \cdot dy \wedge dt)^2. \end{aligned}$$

By opening the brackets we get that

$$(\psi_k^*(\Omega' + \Omega''))^2 = 2\alpha\beta \cdot dx \wedge dy \wedge dz \wedge dt.$$

Last identity means that induced form is non-degenerated. The closedness is implied by the commutation of differential and ψ_k^* . Thus $\psi_k^*(\Omega' + \Omega'')$ is closed and non-degenerate i.e. symplectic form. We proved the first item of the theorem.

Let's prove the second one. Denote by L the bundle defined by multipliers (5). Earlier we noted that theta functions of degree k are sections of $L^{\otimes k}$.

Recall that any complex line bundle over a manifold M is induced by the universal bundle over the complex projective space when M is mapped to \mathbb{CP}^n . Therefore $L^{\otimes k}$ is a pullback of universal bundle and curvature form of $L^{\otimes k}$ is a pullback of the Fubini-Study form, which is a curvature form of universal bundle. Recall also that the first Chern class of line bundle is realized by curvature form. Thus the cohomological class of induced form coincides with $c_1(L^{\otimes k}) = k \cdot c_1(L)$ and we must prove that

$$c_1(L) = [(dz - xdy) \wedge dx + dy \wedge dt].$$

Consider the cover of M_{KT} by sets

$$U_\lambda = \lambda \cdot U_0, \quad \lambda \in \Gamma$$

where $U_0 = \{|u^k| < 3/4\}$. The nerve $N(\mathcal{U})$ of the minimal subcover \mathcal{U} of the above cover U_λ is homeomorphic to M_{KT} and its cohomologies with coefficients in \mathbb{Z} coincide with $H^*(M_{KT}; \mathbb{Z})$.

The coordinate transformations $g_{\lambda\mu} : U_\lambda \cap U_\mu \rightarrow \mathbb{C}^*$ are expressed in terms of multipliers

$$g_{\lambda\mu}(u) = e_\lambda(u) \cdot e_{\mu^{-1}}(\mu \cdot u); \quad \lambda, \mu \in \Gamma. \quad (20)$$

By definition the cocycle $z_{\lambda\mu\nu} \in C^2(\mathcal{U}; \mathbb{Z})$

$$z_{\lambda\mu\nu} = \frac{1}{2\pi i} (\log(g_{\lambda\mu}) + \log(g_{\mu\nu}) - \log(g_{\nu\lambda})) \quad (21)$$

realizes the first Chern class of bundle L . Given formula is the value of z at a two-dimensional simplex $(\lambda, \mu, \nu) \in N(\mathcal{U})$.

The group $H_2(M_{KT}; \mathbb{Z})$ is generated by homological classes of tori T_{ac} , T_{bc} , T_{da} , T_{db} , spanned by commuting translations (1).

Define the functions $f_\lambda(u)$ as follows

$$e_\lambda(u) = e^{2\pi i f_\lambda(u)}. \quad (22)$$

By (20)–(22),

$$c_1([T_{\lambda\mu}]) = f_\mu(u) + f_\lambda(\mu \cdot u) - f_\lambda(u) - f_\mu(\lambda \cdot u).$$

Calculate the first Chern class at the basis tori

$$c_1([T_{ca}]) = c_1([T_{bd}]) = 1; \quad c_1([T_{cb}]) = c_1([T_{ad}]) = 0. \quad (23)$$

Since manifold M_{KT} is a homogenous space of nilpotent Lie group, any element of $H^2(M_{KT}; \mathbb{R})$ is realized by the left-invariant form dual to basis cocycle. Group $H^2(M_{KT}; \mathbb{R})$ is generated by cohomological classes of the forms $(dz - xdy) \wedge dx$, $dy \wedge dt$, $(dz - xdy) \wedge dy$ and $dx \wedge dt$.

By (23), $c_1(L) = [(dz - xdy) \wedge dx + dy \wedge dt]$. Theorem is proved.

References

- [1] L. Auslander, *Lecture notes on nil-theta functions*, Regional Conference Series in Mathematics, **34**. AMS, Providence, R.I., 1977.
- [2] W.D. Kirwin, A. Uribe, *Theta-functions on the Kodaira–Thurston manifold*, [arXiv:0712.4016](#) [math.DG]
- [3] W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer.Math. Soc. **55** (1976), 467–468.
- [4] D. Mumford, *Tata Lectures on Theta I, II*, Birkhäuser, Boston-Basel-Stuttgart, 1983, 1984
- [5] P. Griffiths and J. Harris, *Principles of algebraic geometry*, N.Y. - Chichester - Brisbane - Toronto : John Wiley and Sons, 1978.
- [6] I. A. Taimanov, *Secants of Abelian varieties, theta functions, and soliton equations*, Russian Mathematical Surveys, **52(1)**, (1997), 147–218, [arXiv:alg-geom/9609019](#)

Institute of Mathematics and Informatics, Yakut State University, Yakutsk, Russia.

E-mail: egorov.dima@gmail.com